Third Midterm Review Solutions

Problem 1:
Suppose we have a pair of trick coins as follows. The first coin is fair, but if the first coin shows heads then the second coin automatically will as well, and if the first coin shows tails then the second coin is fair.
1) Compute expected value and covariance matrix of the two random variables.
2) Find linear combinations of the two random variables which are uncorrelated, and compute their variances.
Solution: 1) Assign the value 1 to the outcome of "heads" and 0 to the outcome of tails for each coin. Combine the two random variables into a 2-component random vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

There are only three possible outcomes, which occur with the following probabilities:

- With probability $\frac{1}{2}$, both coins show heads.
- With probability $\frac{1}{4}$, the first coin shows tails and the second shows heads.
- With probability $\frac{1}{4}$, the first coin shows tails and the second shows tails.

The expected value is therefore

$$E[X] = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$$

and the covariance matrix is

$$\begin{aligned} K = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} &= \sum_j p_j[(X_j - \mu)(X_j - \mu)^T] \\ &= \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{4} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{3}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{16} \end{bmatrix} \end{aligned}$$

(2) The eigenvectors of the covariance matrix are uncorrelated random variables that are linear combinations of the originals, and the corresponding eigenvalues are their variances. We therefore diagonalize the covariance matrix as follows. The characteristic polynomial of $\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} = 16K$ is

$$[(4-\lambda)(3-\lambda) - 4] = (\lambda^2 - 7\lambda + 8)$$

And so (using the quadratic formula) the eigenvalues are $\lambda = \frac{7 \pm \sqrt{17}}{2}$. Therefore, the Eigenvalues of K are $\frac{1}{16}$ times these, i.e.

$$\lambda = \frac{7 \pm \sqrt{17}}{32}.$$

A little computation shows the diagonal form is:

$$K = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{16} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(1-\sqrt{17}) & \frac{1}{4}(1+\sqrt{17}) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{32}(7-\sqrt{17}) & 0 \\ 0 & \frac{1}{32}(7+\sqrt{17}) \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{17}} & \frac{1}{34}(17+\sqrt{17}) \\ \frac{2}{\sqrt{17}} & \frac{1}{2}-\frac{1}{2\sqrt{17}} \end{bmatrix}$$

Notice here the change of basis matrices, while they have orthogonal columns, are not orthogonal since we have not normalized the columns. In particular, the first matrix is not the transpose of the third. The computation of this diagonal form is slightly messy: any question on the quiz will be easily computable by hand, so don't worry about this part.

The uncorrelated random variables/eigenvectors are then (now normalizing, which we do by multiplying by the first matrix below)

$$\begin{bmatrix} \frac{1}{|(\frac{1}{4}(1-\sqrt{17}),1)|} & 0 \\ 0 & \frac{1}{|(\frac{1}{4}(1+\sqrt{17}),1)|} \end{bmatrix} \begin{bmatrix} \frac{1}{4}(1-\sqrt{17}) & 1 \\ \frac{1}{4}(1+\sqrt{17}) & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{34-2\sqrt{17}}{16}}\left(\frac{1}{4}(1-\sqrt{17})x_1 + x_2\right) \\ \sqrt{\frac{34+2\sqrt{17}}{16}}\left(\frac{1}{4}(1+\sqrt{17})x_1 + x_2\right) \end{bmatrix}$$

and their variances are the corresponding eigenvalues $\frac{7-\sqrt{17}}{32}, \frac{7+\sqrt{17}}{32}$.

Problem 2:
Consider the data set given by the following points in the xy plane:

$$\{1,1\}, \{2,3\}, \{3,5\}, \{4,7\}$$

.

(1) Compute the covariance matrix by applying singular value decomposition.
(2) Find linear combinations of the variables which are uncorrelated and their variances.
Solution:

We can combine the data into a pair of vectors $\boldsymbol{x}, \boldsymbol{y}$.

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

Which we can combine into a single matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 5 \\ 4 & 7 \end{bmatrix}.$$

Next, we calculate the projection matrix PA by subtracting the mean of each column. The mean of the first column is $\frac{1}{4}(1+2+3+4) = \frac{5}{2}$, and the mean of the second column is $\frac{1}{4}(1+3+5+7) = 4$. Therefore

$$PA = \begin{bmatrix} -\frac{3}{2} & -3 \\ -\frac{1}{2} & -1 \\ \frac{1}{2} & 1 \\ \frac{3}{2} & 3 \end{bmatrix}.$$

If $PA = U\Sigma V^T$ is the singular value decomposition of PA, then the covariance matrix is given by $K = \frac{V\Sigma^T\Sigma V^T}{n-1}$ where n is the number of rows in PA (compare with eq. (314) and (316) in the Lecture notes). The singular value decomposition in this case is

$$\begin{aligned} PA &= \begin{bmatrix} -\frac{3}{2} & -3 \\ -\frac{1}{2} & -1 \\ \frac{1}{2} & 1 \\ \frac{3}{2} & 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{2\sqrt{5}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{38}} & -\frac{3}{2\sqrt{95}} \\ -\frac{1}{2\sqrt{5}} & 0 & 0 & \frac{\sqrt{19}}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & 0 & 3\frac{\sqrt{2}}{\sqrt{19}} & \frac{1}{2\sqrt{95}} \\ \frac{3}{2\sqrt{5}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{38}} & \frac{3}{2\sqrt{95}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \end{aligned}$$

Here we have omitted the computation of the singular value decomposition. As in problem 1, any computations appearing on the quiz will be doable by hand. For practice computing singular value decompositions, see problem 3.

With the above singular value decomposition, we compute:

$$\begin{aligned} K = \frac{V\Sigma^T\Sigma V^T}{n-1} &= \frac{1}{3} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \\ &= \frac{1}{15} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{4} & \frac{5}{2} \\ \frac{5}{2} & 5 \end{bmatrix} \end{aligned}$$

(2) The new variables $\begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' \end{bmatrix} := \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' \end{bmatrix} := \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} V$ will be uncorrelated. (These are the eigenvectors of K, since the above provides an orthonormal diagonalization). In our case, these are

$$\begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \boldsymbol{x} + 2\boldsymbol{y} \\ -2\boldsymbol{x} + \boldsymbol{y} \end{bmatrix}.$$

The corresponding variances are the eigenvalues $\frac{25}{3}$, and 0. Notice that the second combination having zero variance makes sense: all the data points lie on the line 2x-y=1, so the second linear combination is constant for our data set!

Problem 3:
Consider the matrix

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 5 & 2 \end{bmatrix}$$

(1) Compute the Singular Value Decomposition of A.
(2) Write A as a sum of rank 1 matrices.

(3) Compute the Pseudo-Inverse of A.

(4) Find a w such that Aw is closest to $\boldsymbol{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then compute the projection of $\boldsymbol{b}$ onto $C(A)$.

Solution:

(1) We are looking for a decomposition $A = U\Sigma V^T$. Here, U is 2×2 and V is 3×3. First we compute the eigenvectors and eigenvalues of

$$AA^T = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \\ 1 & 12 \end{bmatrix} = \begin{bmatrix} 26 & 2 \\ 2 & 29 \end{bmatrix}.$$

The characteristic polynomial is $(26 - \lambda)(29 - \lambda) - 4 = \lambda^2 - 55\lambda + 750 = (\lambda - 30)(\lambda - 25)$, and so the eigenvalues are $\lambda = 30, 25$. The corresponding eigenvectors are

$$N \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = \mathbb{R} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad N \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \mathbb{R} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Which we normalize to obtain

$$U = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}.$$

Next we find V using the formula $A^T u_i = \sigma_i v_i$. Notice that the singular values are $\sigma = \sqrt{30}, 5$. We obtain

$$A^T u_1 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \\ 1 & 12 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{5}} \\ \frac{10}{\sqrt{5}} \\ \frac{25}{\sqrt{5}} \end{bmatrix} = \sqrt{30} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$A^T u_2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \\ 1 & 12 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{10}{\sqrt{5}} \\ -\frac{5}{\sqrt{5}} \\ 0 \end{bmatrix} = 5 \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}.$$

This gives two of the vectors in the orthonormal basis v_i. To find the third we apply Gram-Schmidt. The vector $\widetilde{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ is easily seen to be linearly uncorrelated from the other two. Thus we compute

$$v_3 = \widetilde{v}_3 - (\widetilde{v}_3 \cdot v_1)v_1 - (\widetilde{v}_3 \cdot v_2)v_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - (-\frac{1}{\sqrt{6}}) \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} - 0 = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \\ -\frac{5}{6} \end{bmatrix}$$

which normalized (still denoting it by the same symbol) is

$$v_3 = \begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ -\frac{5}{\sqrt{30}} \end{bmatrix}$$

(A faster but less general way to compute v_3 in this case is to take $v_3 = v_1 \times v_2$. Of course, this only works for $\mathbb{R}^3$).

The Singular Value Decomposition is therefore (remembering the v_i are the *rows* of V^T)

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}.$$

(2) The Singular Value Decomposition gives A as a sum of the rank 1 matrices that have a single σ_i on the diagonal.

$$\begin{aligned} A &= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}. \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 2\sqrt{5} & \sqrt{5} \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{5} & -\sqrt{5} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \end{bmatrix}. \end{aligned}$$

(3) The Pseudo-Inverse of $A = U\Sigma V^T$ is given by $A^+ = V\Sigma^+ U^T$. Using the matrices calculated above we have

$$\begin{aligned} A^+ = V\Sigma^+ U^T &= \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{5}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{30}} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{5}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{150}} & \frac{2}{\sqrt{150}} \\ \frac{2}{5\sqrt{5}} & -\frac{1}{5\sqrt{5}} \\ 0 & 0 \end{bmatrix} \\ &= \frac{1}{150} \begin{bmatrix} 29 & -2 \\ -2 & 26 \\ 5 & 10 \end{bmatrix}. \end{aligned}$$

(4) A vector w such that Aw is closest to b is given by (see page 87 of Lecture Notes)

$$A^+ \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{150} \begin{bmatrix} 29 & -2 \\ -2 & 26 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{150} \begin{bmatrix} 79 \\ 98 \\ 55 \end{bmatrix}$$

And the projection onto the Column space is $AA^+ \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. This is

$$AA^+ \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 5 & 2 \end{bmatrix} \frac{1}{150} \begin{bmatrix} 79 \\ 98 \\ 55 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

.

Of course this is as expected: since A has full rank, the column space is all of $\mathbb{R}^2$.

Problem 4:

1) Find an orthonormal basis of $\mathbb{R}^2$ in which the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

is diagonal. Is this matrix positive (semi)-definite?
2) Is the matrix

$$\begin{bmatrix} 3 & 7 \\ 7 & -1 \end{bmatrix}$$

positive (semi)-definite?
Solution: (1) To find an orthonormal basis in which the matrix is diagonal, we must decompose $A = QDQ^T$ where Q is an orthogonal matrix. To do this, we diagonalize and orthogonormalize the eigenvectors.

The characteristic polynomial is

$$\det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$$

so $\lambda = 1, 3$ are the eigenvalues. At this point, since the eigenvalues are all strictly positive, we may conclude that the matrix is **positive definite**. Since the eigenvalues are distinct, the eigenvectors are therefore automatically orthogonal so we must find the normalized eigenvector corresponding to each (i.e. no Gram-Schmidt is needed). These are the normalized vectors spanning

$$N \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad\qquad N \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

which are $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively. These vectors form the desired orthonormal basis, and the diagonal form is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(2) A matrix is positive (semi)-definite if and only if it's energy is (respectively) strictly positive, or non-negative. The energy of the matrix is

$$E = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3x + 7y \\ 7x - y \end{bmatrix} = 3x^2 - y^2 + 14xy.$$

This quantity is neither strictly positive nor non-negative since, for instance, $(x, y) = (0, 1)$ results in the value -1. The matrix is therefore neither positive definite nor positive semi-definite.